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Instability of a thermal boundary layer in a constant electric field

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The thermal boundary layer near a heated vertical plate in a poorly conducting liquid is subject to a horizontal d.c. electric field. If the electric field is strong enough, the boundary layer becomes unstable. In this paper a theory is developed to predict the onset of this instability. Experiments measuring the threshold voltage for instability are compared with the theoretical predictions. Other experiments are reported which determine the effect of this instability on the heat transferred from the heated plate.

1. Introduction

Heating a vertical plate immersed in a fluid produces a thermal boundary layer near the plate. This boundary layer may be made unstable by the application of a d.c. electric field perpendicular to the plate. This configuration, which is to be studied in this paper, is shown in figure 1. The other electrode is parallel to the heated plate and outside the boundary layer. The action of the electric field is as follows: the electrical conductivity of the fluid is a function of temperature and thus heating the plate causes gradients in conductivity; free charge accumulates because of the electric field and the conductivity gradients; finally, electrical forces resulting from the action of the electric field on the free charge produce the instability of the boundary layer.

Free convection from a heated vertical plate in ordinary hydrodynamics has received much attention. Ede (1967) presents a review of this subject. Investigations of increased heat transfer in electric fields have been principally concerned with measurements of heat transfer between horizontal cylinders as a function of voltage. Bibliographies of work in this area are contained in Baboi *et al.* (1965) and Turnbull (1969). Kronig & Schwarz (1949) used similarity theory to explain the measurements of increased heat transfer. An approximate solution of the free convection between a horizontal wire and a concentric cylinder with a radial electric field was found by Lykoudis & Yu (1963) by extending Langmuir's conduction model for fine wires. Grosu & Bologa (1968) developed similarity criteria for heat transfer in electric fields. The effect of a d.c. electric field on a stable thermal boundary layer was considered by Turnbull (1969), who showed that the field increases the velocity and decreases the boundary-layer thickness with the effect being larger near the lower edge of the heated plate. Other related work in electrohydrodynamics includes the instability of a fluid heated from above with a vertical d.c. electric field (Turnbull 1968*a*) and the instability of a horizontal fluid surface under the influence of a d.c. field (Melcher 1961).

The organization of this paper is as follows: §2 contains a description of the form of the instability; §3 gives a theoretical prediction of the voltage for incipience of instability; §4 has the results of experimental measurements of the voltage for incipience using visual detection of the instability; and §5 contains the results of measurements of the increased heat transfer due to the boundary-layer instability.

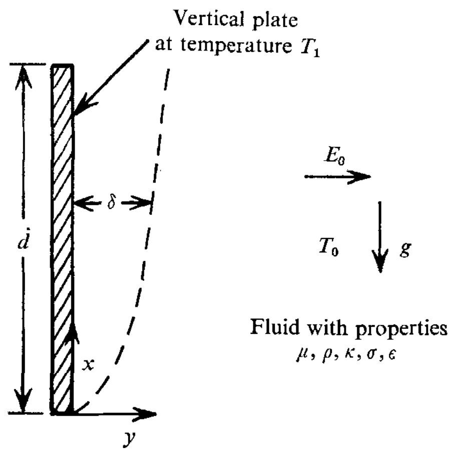

FIGURE 1. Configuration being studied. A heated vertical plate at temperature T_1 is in a liquid at temperature $T_0 < T_1$. There is a thermal boundary layer near the plate and a horizontal electric field is applied to the liquid.

2. Description of the instability

Figure 2 (plate 1) contains photographs illustrating various stages of the instability. The pictures were taken using a parallel light source set so that the light travelled horizontally parallel to the heat plate. The camera then captured the resulting diffraction pattern. A mask is used so that only light passing through the boundary layer reaches the camera. Figure 2 (*a*) is the picture for zero voltage and shows no perturbation since the boundary layer is stable in the absence of an electric field.

At a low voltage, the picture is the same as for no voltage. As the voltage is raised past a threshold, the first manifestation of the instability appears. This instability appears as perturbations in the boundary layer, which move upward with the fluid velocity. The diffraction pattern for the incipient instability is shown in figure 2 (*b*). As the voltage is again raised, the instabilities become larger until the boundary layer is destroyed and cells form between the electrodes. The unstable boundary layer is shown in figure 2 (*c*). The instability now has such a large growth rate that it destroys the boundary layer before the fluid motion can sweep it out of the system. Finally, for still higher voltages, the entire fluid becomes turbulent, as shown in figure 2 (*d*).

3. Prediction of the incipience of instability

In this section, an approximate analysis will be developed which permits a prediction of the voltage for incipience of instability in the boundary layer. The configuration is that of figure 1 with a vertical velocity in the boundary layer near the heated plate and a horizontal electric field. The other electrode which is parallel to the heated plate and outside the boundary layer is not shown in the figure. The fluid is assumed to be slightly conducting and to obey Ohm's law. Since the electrical conductivity of slightly conducting liquids is a function of temperature, there are gradients in conductivity in the boundary layer. When an electric field is applied, free charge results, producing a force equal to the charge density times the electric field. The effect of this force is similar to the effect of the gravitational force in the Bénard problem. The Bénard problem consists of a stationary horizontal fluid layer heated from below, thus making the fluid lighter on the bottom. When the temperature gradient exceeds a critical value, the gravitational forces cause the fluid to become unstable and convection results.

The gravitational force in this problem is perpendicular to the electrical force and is balanced by the viscous force resulting from the velocity in the boundary layer. Since both the gravitational force and the velocity in the boundary layer are perpendicular to the electrical forces and since the boundary layer is stable when the voltage is zero, it will be assumed that the instability is caused purely by electrical forces and that gravity and the equilibrium velocity have no effect on the threshold for instability. The velocity has an effect once an instability forms. This effect is to carry the instability upward and out of the system.

In addition to the assumptions previously discussed, the fluid is assumed to be incompressible and the variation in boundary-layer thickness with height is neglected. The current in the fluid is assumed to obey Ohm's law with the conductivity a function of temperature. Although not always valid for poorly conducting liquids, Ohm's law has been used successfully to predict motions in a class of liquids (Melcher & Firebaugh 1967; Turnbull 1968*b*). The model to be tested for stability is then a stationary liquid with a temperature gradient in a thin region next to a plate. An electric field perpendicular to the plate provides the only body force on the liquid. For this approximate analysis the only fluid property whose variation with temperature is considered is the electrical conductivity. The mechanical equations are then

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + q\mathbf{E}, \tag{1}$$

$$\nabla . \mathbf{v} = 0, \tag{2}$$

where ρ is the mass density, $D/Dt = \partial/\partial t + (\mathbf{v} . \nabla)$ is the convective derivative, $\mathbf{v}$ is the perturbation velocity, p the pressure, μ the viscosity, q the charge density and $\mathbf{E}$ the electric field.

In a poorly conducting liquid, the currents are small and, therefore, magnetic fields are negligible giving as the electric equations

$$q = \nabla . (\epsilon \mathbf{E}), \tag{3}$$

$$\nabla \times \mathbf{E} = 0, \tag{4}$$

$$\frac{\partial q}{\partial t} + \nabla . \mathbf{J} = 0, \tag{5}$$

$$\mathbf{J} = \sigma \mathbf{E} \times q\mathbf{v}, \tag{6}$$

where ϵ is the dielectric constant, $\mathbf{J}$ the current density, and σ the electrical conductivity. Of the electrical properties of the fluid, conductivity and permittivity, the conductivity is a much stronger function of temperature in slightly conducting liquids. In the steady state with no convection, (5) and (6) yield

$$\nabla . \mathbf{E} = -\mathbf{E} . \nabla\sigma/\sigma. \tag{7}$$

This gives, with (3),

$$q = \mathbf{E} . \nabla\epsilon - (\epsilon/\sigma)\mathbf{E} . \nabla\sigma. \tag{8}$$

The ratio of the two terms in (8) is $(\mathbf{E} . \nabla\epsilon/\epsilon)/(\mathbf{E} . \nabla\sigma/\sigma)$, which has a magnitude much less than one. Therefore, the gradients in dielectric constant have a negligible effect on the charge and electric field distribution. The ratio of the force due to the gradient in dielectric constant to the free charge force is

$$\frac{-\frac{1}{2}\mathbf{E} . \mathbf{E}\nabla\epsilon}{q\mathbf{E}} = \frac{\frac{1}{2}\mathbf{E} . \mathbf{E}\nabla\epsilon}{E(\epsilon/\sigma)(\mathbf{E} . \nabla\sigma)}. \tag{9}$$

Again, the ratio is much less than one and the gradients in dielectric constant can be neglected in the force equation also. The preceding proof is given in more detail in Turnbull (1967) without the steady-state restriction.

To complete the set of equations, the equations of heat conduction and that of the functional dependence of the electrical conductivity on temperature are needed. These relations are

$$DT/Dt = \kappa\nabla^2 T, \tag{10}$$

$$\sigma = \sigma_0[1 + \alpha(T - T_0)], \tag{11}$$

where κ is the thermal diffusivity.

The stability of the boundary layer will be determined by assuming perturbations from an equilibrium and by testing whether these perturbations grow or decay in time. The equilibrium state is assumed to consist of a temperature gradient in a motionless fluid, i.e. the steady flow in the vertical direction is neglected for this calculation. The equilibrium temperature varies from T_1 at the heated plate to T_0 at the edge of the boundary layer. The boundary-layer thickness, δ, is assumed to be constant. In reality, however, δ varies in the vertical direction. In the equilibrium state the electrical force is balanced by the pressure.

The system is now assumed to be perturbed from equilibrium. The perturbation variables are the electric potential ϕ, charge density q', pressure p', velocity $\mathbf{v}$, temperature T', and electrical conductivity σ'. The perturbation electric field is related to the potential by

$$\mathbf{E}' = -\nabla\phi. \tag{12}$$

The perturbations are assumed to be of the form

$$q' = \mathrm{Re}[\hat{q}'(y)\, e^{j(\omega t - k_x x - k_z z)}]. \tag{13}$$

The perturbation variables are substituted into (1)–(6), and (10)–(11). These equations are then linearized with respect to the perturbation variables and the following equations result:

$$j\omega\rho\hat{v}_x = jk_x\hat{p}' + \mu(D^2 - k^2)\hat{v}_x + qjk_x\hat{\phi}, \tag{14}$$

$$j\omega\rho\hat{v}_y = -D\hat{p}' + \mu(D^2 - k^2)\hat{v}_y - qD\hat{\phi} + E_y\hat{q}', \tag{15}$$

$$j\omega\rho\hat{v}_z = jk_z\hat{p}' + \mu(D^2 - k^2)\hat{v}_z + qjk_z\hat{\phi}, \tag{16}$$

$$-jk_x\hat{v}_x - jk_z\hat{v}_z + D\hat{v}_y = 0, \tag{17}$$

$$q = -\epsilon(D^2 - k^2)\hat{\phi}, \tag{18}$$

$$j\omega\hat{q}' - \sigma(D^2 - k^2)\hat{\phi} + \hat{\sigma}'DE_y - D\sigma D\hat{\phi} + E_yD\hat{\sigma}' + \hat{v}_yDq = 0, \tag{19}$$

$$j\omega\hat{T}' + \hat{v}_yDT = \kappa(D^2 - k^2)\hat{T}', \tag{20}$$

$$\hat{\sigma}' = \sigma_0\alpha\hat{T}', \tag{21}$$

where $k^2 = k_x^2 + k_z^2$ and $D = \partial/\partial y$.

Equations (14) to (18) combined into a single equation in $\hat{v}_y$ and $\hat{\phi}$ yield

$$j\omega\rho(D^2 - k^2)\hat{v}_y = \mu(D^2 - k^2)\hat{v}_y - k^2\epsilon D^2E_y\hat{\phi} + k^2\epsilon E_y(D^2 - k^2)\hat{\phi}. \tag{22}$$

Combining (18), (19) and (21) gives

$$-(\sigma + j\omega\epsilon)(D^2 - k^2)\hat{\phi} - D\sigma D\hat{\phi} + \epsilon D^2E_y\hat{v}_y + \sigma_0\alpha E_yD\hat{T}' + \sigma_0\alpha DE_y\hat{T}' = 0. \tag{23}$$

Since the instability predicted by (20), (22) and (23) is similar to the Bénard problem, approximations made in that problem will be made here. The temperature gradients will be assumed to be small so that the gradients in electrical conductivity are also small. With this approximation, we may neglect $\hat{\phi}D^2E_y$ compared with $E_yD^2\hat{\phi}$, $DE_y\hat{T}'$ compared with $E_yD\hat{T}'$ and $D\sigma D\hat{\phi}$ compared with $\sigma D^2\hat{\phi}$. This is similar to the 'Boussinesq approximation'. In addition, the principle of exchange of stabilities is assumed to hold, i.e. the system becomes unstable at zero frequency.

The equations at incipience of instability ($\omega = 0$) are

$$\hat{v}_yDT = \kappa(D^2 - k^2)\hat{T}', \tag{24}$$

$$\mu(D^2 - k^2)^2\hat{v}_y + k^2\epsilon E_y(D^2 - k^2)\hat{\phi} = 0, \tag{25}$$

$$\sigma(D^2 - k^2)\hat{\phi} = \epsilon D^2E_y\hat{v}_y + \alpha\sigma_0E_yD\hat{T}'. \tag{26}$$

The above equations require a knowledge of the equilibrium temperature distribution. Due to the equilibrium velocity, which we have neglected, the temperature gradient varies across the boundary layer, being largest near the plate. For the purposes of an approximate analysis the temperature gradient will be assumed uniform. This allows (24) to (26) to be combined into a single equation,

$$\frac{\mu\kappa}{DT}(D^2 - k^2)^3\hat{T}' + \frac{k^2\epsilon^2E_y(D^2E_y)\kappa}{\sigma DT}(D^2 - k^2)\hat{T}' + k^2\epsilon\alpha E_y^2D\hat{T}' = 0. \tag{27}$$

Non-dimensionalized, this equation becomes

$$(D^{*2}-k^{*2})^3\hat{T}'+k^{*2}(Rx)\,(Er)\,(D^{*2}-k^{*2})\hat{T}'+k^{*2}\,Er\,D^*\hat{T}' = 0, \tag{28}$$

where $D^* = \delta D$, $k^* = \delta k$, $Rx = \kappa Dq/\alpha\sigma_0(T_1-T_0)E_y$, $Er = \alpha\epsilon E_y^2(T_1-T_0)\delta^2/\mu\kappa$ and δ is the boundary-layer thickness. Er is similar to the Rayleigh number,

$$Ra = g(d\rho/dT)\,(T_1-T_0)\delta^3/\mu\kappa,$$

with the electrical forces replacing the gravitational ones. Rx expresses the importance of convection of charge and is the electrical relaxation time divided by the thermal diffusion time.

For the experiments reported in this paper, the second term in (28) is negligible compared to the third term. For this case, (28) reduces to

$$(D^{*2}-k^{*2})^3\hat{T}'+k^{*2}\,Er\,D^*\hat{T}' = 0. \tag{29}$$

This equation differs from that of the Bénard problem in that the second term has T' replaced by its derivative.

To solve the exact problem for the stability of the boundary layer, the equilibrium velocity and effects of gravity should be added to (24) to (26). The boundary conditions are that the perturbation velocity, potential and temperature vanish at the heated plate and also at the other electrode. In addition, of course, the equilibrium equations would have to be solved first to determine the equilibrium. For the purposes of an approximate solution we will use (29) and assume, as did Lord Rayleigh (1916), in the Bénard problem, that a derivative in the y direction is of magnitude π times the quantity differentiated. That is to say, if the perturbations vary sinusoidally with y, a half wavelength fits within the boundary layer. In this case the minimum Er occurs with $k = \pi\sqrt{2}$ and is $Er = 209$. Since Er is proportional to δ^2, the instability will occur first where the boundary layer is thickest, which is at the top of the heated plate.

4. Experiments to detect the onset of instability

In order to evaluate the theory developed in the last section, experiments were performed to detect the onset of instability. The onset of instability is that point where waves first start to appear in the boundary layer, as in figure 2(*b*). The apparatus used is that described in the introduction. By shining a parallel light through the boundary layer and projecting the resulting diffraction on a screen, the perturbations in the boundary layer were made visible. Starting at zero, the voltage was raised slowly until the perturbation appeared on the screen. The voltage at that point was then recorded. The results of these measurements are shown in figure 3 for measurements taken using corn oil in a tank about 5 in. high, 4 in. across, and with a plate separation of 1 in. The critical electrical Rayleigh number Er would appear to be about 300 to 500 regardless of the heat flux or temperature of the heated plate. For the experiments Rx is of order 0·01 so that the second term in (28) may be neglected. Using the theory of Turnbull (1969) it is found that the electric field at the observed threshold for instability has altered the equilibrium boundary layer only in a region a few millimetres high at the bottom of the heated plate. Thus the value for δ used in calculating Er is that predicted by the theory of Squire (1938). The value of δ used is that at the top of the plate where it is largest. In figure 3, the electric Rayleigh number is plotted against Prandtl number with the fluid properties being taken at the average temperature. The exact point of instability was hard to detect since the perturbations were carried out of the system by the fluid motion. If the growth rate were small and there were no initial disturbance, the instability would not be seen even though the fluid was in an unstable situation. For this reason the

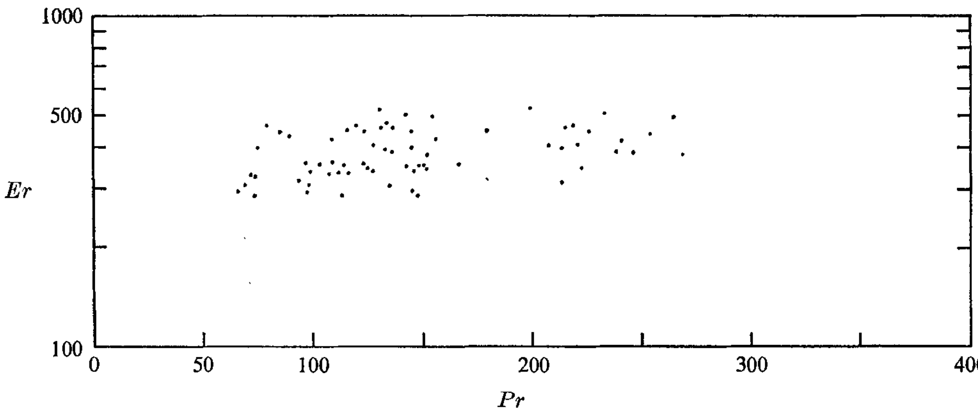

FIGURE 3. Electric Rayleigh number, Er, for incipient instability as a function of the Prandtl number, Pr.

lower values of the measured electric Rayleigh numbers are probably the correct ones. The measured values are to be compared to the theoretically predicted value of about 200. If the effects of the boundary conditions at the plate were fully taken into account, it seems likely that the predicted value would be somewhat greater than 200. The approximate agreement between theory and experiment suggests that the mechanism indicated by the theory is the actual mechanism for instability. That mechanism is: the electrical forces on the free charge tend to produce an instability; the instability is slowed by the viscosity; and the thermal conductivity tends to restore the fluid to its original condition.

5. Heat transfer experiments

If the electric field causes the boundary layer to become unstable, there would be a change in the heat transferred from the heated plate to the other electrode. The apparatus used was the same one as for the previous experiments with the experiments performed as follows: the voltage was set to the desired value and then the heater was turned on with the power into the heater held constant. After the initial transient was over, the temperature of the two plates was measured as a function of time. These measurements were used to calculate the electric Rayleigh number and the Nusselt number based on the height of the plate. Since heat was also transferred away from the side of the heated plate not facing the other electrode, this Nusselt number is meaningful only for comparing the measurements with each other.

The data for zero voltage and a particular heater power were assumed to obey the curve $Nu = C_1 Ra^{\frac{1}{4}} + C_2$ where C_1 and C_2 are arbitrary constants to be determined by a least squares fit and Ra is the ordinary Rayleigh number. This curve fits all data points within 5 %, and most points within 2 %. This form of curve was chosen because heat transfer in free convection is often approximately proportional to $Ra^{\frac{1}{4}}$. The data for non-zero voltages were analysed as follows: the Nusselt number predicted by the curve for zero voltage was subtracted from the value for non-zero voltage. This gives the change in heat transfer due to the electric field. The results are shown in figure 4 as a function of the electric Rayleigh number.

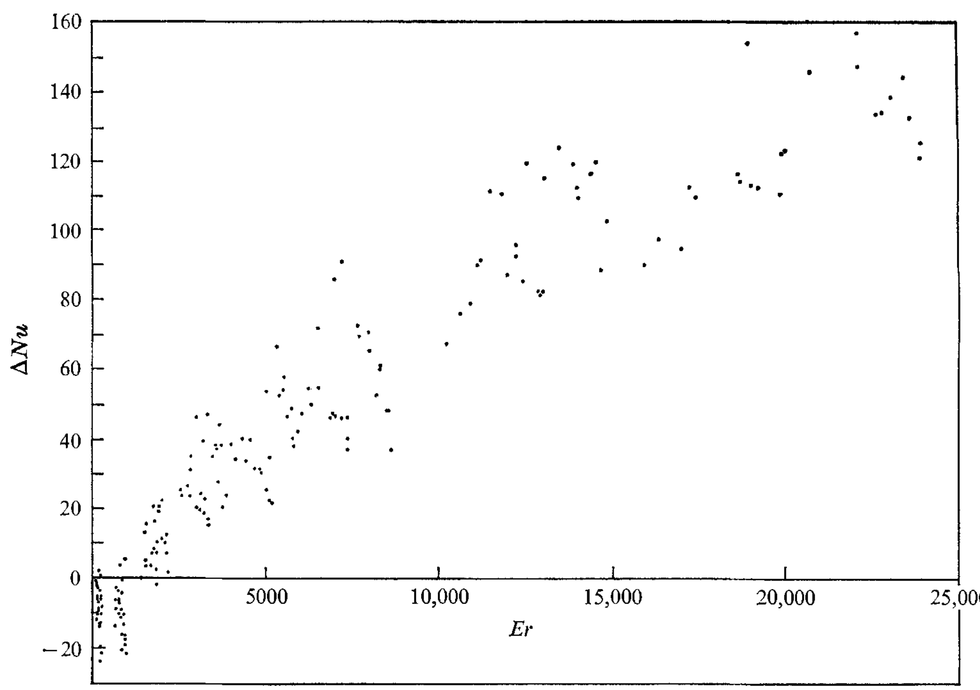

FIGURE 4. Change in Nusselt number, ΔNu, due to the electric field as a function of the electric Rayleigh number, Er.

For small electric Rayleigh numbers, it appears that the field retards the heat transfer slightly. Then at an electric Rayleigh number of about 1000 to 2000 the heat transfer begins to increase and continues to increase with Er. The maximum voltage used was 10 kV. The value of Er for which the Nusselt number begins to increase is somewhat higher than the value for incipient instability. This indicates that the waves in the boundary layer do not affect the heat transfer much and that the rise in Nusselt number occurs due to the breakup of the boundary layer observed at higher voltages.

6. Summary

The electrical forces producing the instability result from the electric field acting on the free charge density. The source of the free charge is as follows: the electrical conductivity is a function of temperature and, therefore, there are conductivity gradients in the boundary layer. A current flows in the fluid because of the electric field and in order for the current to be continuous the electric field must be smallest where the conductivity is largest. The free charge accumulates in order to provide the necessary electric field distribution.

The reason a threshold for instability exists is that the viscosity of the liquid opposes the tendency of the electric field to produce instability while the thermal conductivity acts to destroy the perturbations. Just above the threshold the instability occurs as waves in the boundary layer which are carried out of the system by the upward velocity. At a higher voltage the entire boundary layer becomes unstable and breaks up. Increases in heat transfer are only substantial after the boundary layer has broken up.

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REFERENCES

Baboi, N. F., Baloga, M. K. & Semenov, K. N. 1965 The influence of electric fields on heat transfer in liquids and gases. *Elektronnaya Obrabotka Materialov*, no. 1, 57.

Ede, A. J. 1967 Advances in free convection. In *Advances in Heat Transfer* (ed. T. R. Irvine), vol. 4, 1. Academic.

Grosu, F. P. & Bologa, M. K. 1968 Similarity criteria for convection heat exchange in an electric field. *Elektronnaya Obrabotka Materialov*, no. 2, 50.

Kronig, R. & Schwarz, N. 1949 On the theory of heat transfer from a wire in an electric field. *Appl. Sci. Res.* A **1**, 35.

Lykoudis, P. S. & Yu, C. P. 1963 The influence of electrostrictive forces in natural thermal convection. *Int. J. Heat Mass Transfer*, **6**, 853.

Melcher, J. R. 1961 Electrohydrodynamic and magnetohydrodynamic surface waves and instabilities. *Phys. Fluids*, **4**, 11.

Melcher, J. R. & Firebaugh, M. S. 1967 Travelling-wave bulk electro-convection induces across a temperature gradient. *Phys. Fluids*, **11**, 2597.

Rayleigh, Lord 1916 On convective currents in a horizontal layer of fluid when the higher temperature is on the underside. *Phil. Mag.* **32**, 529.

Squire, H. B. 1938 In *Modern Development in Fluid Dynamics*, p. 641. Oxford University Press.

Turnbull, R. J. 1967 Electroconvective instability with a stabilizing temperature gradient. Ph.D. thesis. Electrical Engineering Dept., M.I.T.

Turnbull, R. J. 1968*a* Electroconvective instability with a stabilizing temperature gradient. I. Theory. *Phys. Fluids*, **11**, 2588.

Turnbull, R. J. 1968*b* Electroconvective instability with a stabilizing temperature gradient. II. Experimental results. *Phys. Fluids*, **11**, 2597.

Turnbull, R. J. 1969 Free convection from a heated vertical plate in a direct-current electrical field. *Phys. Fluids*, **12**, 2255.

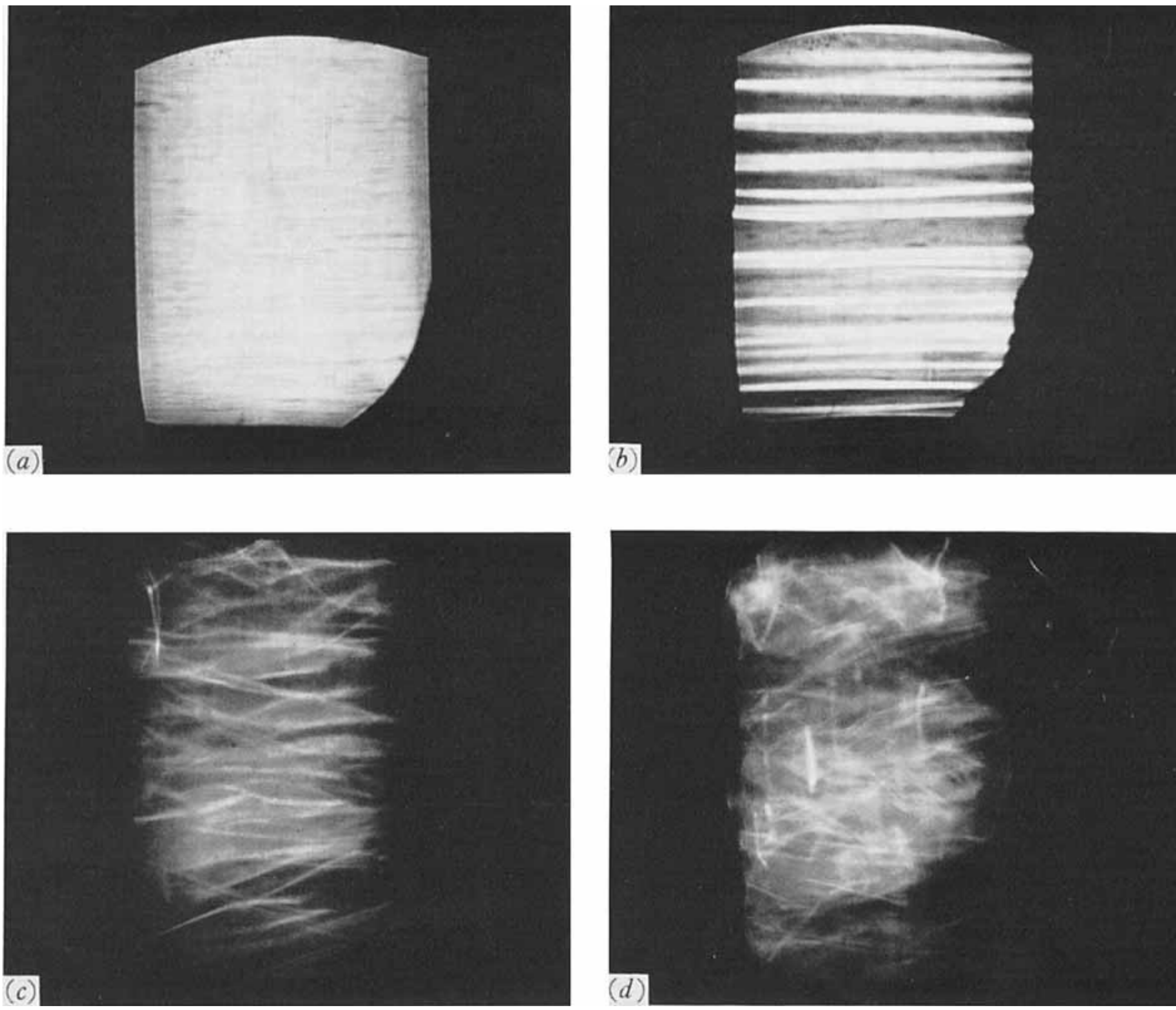

FIGURE 2. The diffraction pattern resulting from parallel light being shone through the boundary layer. (*a*) No electric field. (*b*) Electric field large enough to produce waves in the boundary layer. The horizontal lines are due to perturbations in the boundary layer and are moving up along the plate. (*c*) Electric field large enough to destroy the boundary layer. (*d*) Still larger electric field.